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Extremal Spline Bases

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In the space of all polynomial splines on an infinite equidistant grid with fixed odd degree and with fixed period, the Lagrangians belonging to the grid-points as nodes form an extremal basis with respect to the supremum-norm. This means that no element of the basis can be approximated within the span of the other elements better than by zero. The result is carried over to the nonperiodic cardinal-spline case. Moreover, an intrinsic insight into the behaviour of the Lagrangian splines is obtained.

1. INTRODUCTION AND STATEMENTS OF THEOREMS

In [3] we introduced our concept of extremal bases in normed vector spaces. Such a basis consists of elements which cannot be approximated by linear combinations of the other ones better than by zero. Hence it consists of "outmost linearly independent" elements and furnishes (in some sense) a most stable representation of the space elements by their set of coefficients.

We also investigated conditions, where a Lagrangian basis is extremal with respect to the supremum-norm. This is the case if the norm of all Lagrangians is exactly one, and hence minimal. For instance, the unique extremal basis in the space of polynomial functions on $[-1, 1]$ of degree n consists of the Lagrangians with respect to the Fekete-points. In case of trigonometric polynomials, an extremal basis is obtained by choosing equidistant nodes. In this case, and apart from a constant factor, the Lagrangians are shifted Dirichlet-kernels.

In what follows, we are concerned with polynomial splines of odd degree $m = 2r + 1$ on the grid $\mathbb{Z}$. First let S_m^∞ denote the space of all bounded functions $s \in C_{m-1}(\mathbb{R})$ which aggree with a polynomial of degree m on every interval of the form $[j, j+1]$, $j \in \mathbb{Z}$. The space is provided with the supremum-norm

$$\|s\| := \sup\{|s(x)|: x \in \mathbb{R}\}.$$

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In addition let S_m^N, $N \in \mathbb{N}$, be defined to be the subspace of S_m^∞ consisting of the elements with period N. Then, inspired by the trigonometric case, we may hope that in all spaces S_m^N, $N \in \mathbb{N} \cup \{\infty\}$, the Lagrangians belonging to the grid-points as the interpolation nodes perform an extremal basis.

For $N \in \mathbb{N}$, these Lagrangians are the elements $l_j^{(N)} \in S_m^N$, $j \in \{0, 1, ..., N-1\}$, defined by

$$l_j^{(N)}(k) = \delta_{j,k}, \qquad k \in \{0, 1, ..., N-1\},$$

δ_{jk} denoting Kronecker's symbol. For $N = \infty$, they are the elements $l_j^{(\infty)} \in S_m^\infty$, $j \in \mathbb{Z}$, with

$$l_j^{(\infty)}(k) = \delta_{j,k}, \qquad k \in \mathbb{Z}.$$

THEOREM 1. *Let $m \in \mathbb{N}$ be odd. Then the Lagrangian splines $\{l_j^{(\infty)} \mid j \in \mathbb{Z}\}$ and $\{l_j^{(N)} \mid j = 0, 1, ..., N-1\}$ perform an extremal basis for S_m^∞ and S_m^N, $N \in \mathbb{N}$, respectively.*

By the arguments used by us in [3], Theorem 1 is an immediate consequence of

THEOREM 2. *Let $m \in \mathbb{N}$ be odd. Then $\| l_j^{(N)} \| = 1$ for $N = \infty$ and all $j \in \mathbb{Z}$ and for $N \in \mathbb{N}$ and $j \in \{0, 1, ..., N-1\}$.*

The proof of Theorem 2 will be performed by the use of results and ideas of Meinardus and Merz [1] and of ter Morsche [2]. The qualitative behaviour of periodic Lagrangian splines has been discussed already by Richards [4], in case $m = 3$ and $m = 5$ also by Schurer [5, 6], but no proof of Theorem 2 seems to be known. Moreover, we shall obtain a more intrinsic insight into the behaviour of the Lagrangian splines, which is as in the figures in Section 3.

2. PROOF OF THEOREM 2

If $m = 1$, then the statements of Theorem 2 are obviously true.

Next let $m = 2r + 1$, $r \in \mathbb{N}$. Because of $l_j^{(N)}(t) = l_0^{(N)}(j + t)$ it suffices to prove the statements for $j = 0$.

To begin with, let us consider the periodical case where $N \in \mathbb{N}$. Let

$$q_j^{(N)}(t) := l_0^{(N)}(j + t) \tag{1}$$

for $t \in \mathbb{R}$, $j \in \mathbb{Z}$. By the uniqueness of interpolating periodic splines, we have

$$q_{N-1-j}^{(N)}(t) = q_j^{(N)}(1 - t) \tag{2}$$

for $t \in \mathbb{R}$, $j \in \mathbb{Z}$. Following Richards [4], the $q_j^{(N)}(t)$ have fixed sign for $t \in [0, 1]$, where

$$|q_j^{(N)}(t)| = (-1)^j q_j^{(N)}(t)$$

$$\text{for } \begin{cases} j = 0, 1,..., N-1, & \text{if } N \text{ is odd,} \\ j = 0, 1,..., [(N-1)/2], & \text{if } N \text{ is even.} \end{cases} \tag{3}$$

On the other hand, counting the zeros of the derivatives, we can easily prove that $l_0^{(N)}(t) = q_0^{(N)}(t)$ attains the value one in $[0, 1]$ only once, namely, for $t = 0$, while it vanishes for $t = 1$. Hence we obtain from (2) and (3) the estimate

$$0 \leqslant q_0^{(N)}(t) = q_{N-1}^{(N)}(1-t) \leqslant 1 \qquad \text{for } t \in [0, 1]. \tag{4}$$

The problem now is whether any other $q_j^{(N)}(t)$ exceeds one in absolute value for $t \in [0, 1]$.

Let

$$H_m(t, z) := (1-z)^{m+1} \left(t + z\frac{\partial}{\partial z}\right)^m \frac{1}{1-z} = (1-z)^{m+1} \sum_{v=0}^{\infty} (t+v)^m z^v$$

denote the generalized Euler–Frobenius polynomial of degree m in each variable. Then, following Meinardus and Merz [1], we have

$$q_j^{(N)}(t) = \frac{1}{N} \sum_{\mu=0}^{N-1} \frac{H_m(t, \zeta^\mu)}{H_m(1, \zeta^\mu)} \zeta^{-(j+1)\mu} \tag{5}$$

with

$$\zeta = e^{2\pi i/N}$$

for $t \in [0, 1]$, and due to the definition of H_m, the identities

$$z^m H_m(t, z^{-1}) = H_m(1-t, z), \tag{6}$$

$$z H_m(1, z) = H_m(0, z) \tag{7}$$

hold. In addition it follows from $H_m(t, 0) = t^m$ and (6) that

$$H_m(t, z) = (1-t)^m z^m + \text{terms of lower degree in } z. \tag{8}$$

Due to ter Morsche [2, Theorem 3.1], the roots of $H_m(t, z)$ with respect to z for fixed $t \in [0, 1]$ are real, nonpositive and simple, while $H_m(0, z)$ vanishes for $z = 0$ by (7). Let $H_m(0, z)$ have the roots

$$z_1 < z_2 < \cdots < z_m = 0.$$

By (7) and (6), $H_m(1, z)$ possesses the roots $z_1,..., z_{m-1}$ and $z_0 := -\infty$, while $z_\mu = z_{m-\mu}^{-1}$ for $\mu = 0, 1,..., m$ is valid. Hence, recalling that $m = 2r + 1$, the order of the roots is more precisely

$$z_1 < \cdots < z_r < -1 < z_{r+1} < \cdots < z_{2r} < z_{2r+1} = 0. \tag{9}$$

Now, as remarked by Meinardus and Merz [1, p. 192], the formulation being not quite correct, the roots of $H_m(0, z)$ are separated by those of $H_m(t, z)$ for $0 < t < 1$ (it can be proved that the roots of $H_m(t, z)$, $0 \leqslant t \leqslant 1$, are strictly decreasing). From this and (8) it follows that

$$\frac{H_m(t, z_\mu)}{H_m^z(0, z_\mu)} > 0 \qquad \text{for } \mu = 1, 2,..., m \tag{10}$$

holds for $0 < t < 1$, if H_m^z denotes the partial derivative of H_m with respect to the second argument.

By (9), $H_m(1, z)$ has no root on the unit circle (which is not true if m is even). Besides, the residual of

$$\sum_{\nu=0}^{N-1} \frac{1}{z - \zeta^\nu} = \frac{Nz^{N-1}}{z^N - 1}$$

at ζ^μ, $\mu\{0, 1,..., N-1\}$, is one. Hence we obtain from (5) the representation

$$q_j^{(N)}(t) = \frac{1}{2\pi i} \int_{\mathfrak{C}_1 - \mathfrak{C}_2} \frac{H_m(t, z)}{H_m(1, z)} \cdot \frac{z^{N-j-1}}{z^N - 1} \cdot \frac{dz}{z}$$

if $\mathfrak{C}_1$ and $\mathfrak{C}_2$ are positively orientated circles with zero as center and radius ρ_1 and $\rho_2 = \rho_1^{-1}$, respectively, where $\rho_1 > 1$ is sufficiently small.

Replacing z by z^{-1} and by the use of (6) we obtain

$$\int_{-\mathfrak{C}_2} \frac{H_m(t, z)}{H_m(1, z)} \cdot \frac{z^{N-j-1}}{z^N - 1} \cdot \frac{dz}{z} = \int_{\mathfrak{C}_1} \frac{H_m(1-t, z)}{H_m(0, z)} \cdot \frac{z^j}{z^N - 1} \cdot \frac{dz}{z}.$$

Inserting this above we get finally by the use of (7)

$$q_j^{(N)}(t) = \frac{1}{2\pi i} \int_{\mathfrak{C}_1} \frac{H_m(t, z)\, z^{N-j-1} + H_m(1-t, z)\, z^j}{H_m(0, z)(z^N - 1)}\, dz \tag{11}$$

for $t \in [0, 1]$, $j \in \{0, 1,..., N-1\}$.

The singulatities of the integrand of (11) in the outer region of $\mathfrak{C}_1$ are located at $z_1,..., z_r$ and possibly at infinity, but nowhere else. It follows from

(8) that the residual at infinitely is $-(1-t)^m$ if $j=0$, $-t^m$ if $j=N-1$ and zero else. Since $z_1,\dots,z_r$ are poles of the first order, we therefore obtain from (11) the fundamental representation

$$q_j^{(N)}(t)=(1-t)^m\,\delta_{0,j}+t^m\delta_{N-1,j} - \sum_{\mu=1}^{r}\frac{H_m(t,z_\mu)\,z_\mu^{N-j-1}+H_m(1-t,z_\mu)\,z_\mu^j}{H_m^z(0,z_\mu)(z_\mu^N-1)} \tag{12}$$

for $t\in[0,1]$, $j\in\{0,1,\dots,N-1\}$.

As a first consequence of (12), the limit

$$q_j^{(\infty)}(t):=\lim_{N\to\infty}q_j^{(N)}(t)=(1-t)^m\,\delta_{0,j}-\sum_{\mu=1}^{r}\frac{H_m(t,z_\mu)\,z_\mu^{-j-1}}{H_m^z(0,z_\mu)} \tag{13}$$

exists for $t\in[0,1]$, $j\in\mathbb{N}_0$. Note that it follows from (4) that

$$0\leqslant q_0^{(\infty)}(t)\leqslant 1 \qquad \text{for } 0\leqslant t\leqslant 1 \tag{14}$$

is valid, while (3) yields together with (10)

$$(-1)^j\,q_j^{(\infty)}(t)=(1-t)^m\,\delta_{0,j}+\sum_{\mu=1}^{r}\frac{H_m(t,z_\mu)}{H_m^z(0,z_\mu)}\,|z_\mu|^{-j-1}\geqslant 0.$$

Hence we obtain by (9) and (14) the inequalities

$$1\geqslant q_0^{(\infty)}(t)\geqslant -q_1^{(\infty)}(t)\geqslant q_2^{(\infty)}(t)\geqslant -q_3^{(\infty)}(t)\geqslant\cdots\geqslant 0. \tag{15}$$

for arbitrary $t\in[0,1]$.

The convergence in (13) is uniform for all $t\in[0,1]$ and any finite set of $j-s$. This means that $l_0^{(N)}(t)$ converges uniformly on every compact set in $\mathbb{R}$, and this to an element of S_m^∞. Because of $l_0^{(N)}(j)=0$ for $j\in\mathbb{Z}\backslash\{0\}$, this element must be the Lagrangian cardinal spline $l_0^{(\infty)}$, defined above, where

$$l_0^{(\infty)}(j+t)=q_j^{(\infty)}(t) \qquad \text{for } t\in[0,1],\ j\in\mathbb{N}_0.$$

This, together with $l_0^{(\infty)}(-x)=l_0^{(\infty)}(x)$ for $x\in\mathbb{R}$ and with (15), yields

$$\|l_0^{(\infty)}\|=1,$$

as the assertion of Theorem 2 is.

Next let $N\in\mathbb{N}$ be even. Then we obtain from (12), (3) and (10)

$$0\leqslant(-1)^j\,q_j^{(N)}(t) =(1-t)^m\,\delta_{0,j}+\sum_{\mu=1}^{r}\frac{H_m(t,z_\mu)\,|z_\mu|^{N-j-1}-H_m(1-t,z_\mu)\,|z_\mu|^j}{H_m^z(0,z_\mu)(|z_\mu|^N-1)}$$

for $t \in [0, 1]$, $j \in \{0, 1, ..., [(N-1)/2]\}$. Hence it follows that

$$(-1)^{j-1} q_{j-1}^{(N)}(t) - (-1)^j q_j^{(N)}(t)$$
$$= (1-t)^m \delta_{0,j-1} + \sum_{\mu=1}^{r} \frac{H_m(t, z_\mu)\,|z_\mu|^{N-j-1} + H_m(1-t, z_\mu)\,|z_\mu|^{j-1}}{H_m^z(0, z_\mu)(|z_\mu|^N - 1)}$$
$$\times (|z_\mu| - 1) \geqslant 0$$

and, because of (4), that

$$0 \leqslant \cdots \leqslant (-1)^j q_j^{(N)}(t) \leqslant (-1)^{j-1} q_{j-1}^{(N)} \leqslant \cdots \leqslant q_0^{(N)}(t) \leqslant 1 \tag{16}$$

holds for $t \in [0, 1]$, $j \in \{1, 2, ..., [(N-1)/2]\}$. From this the assertion of Theorem 2 follows in view of (2).

Finally let N be odd. In this case it follows from (3), (10) and (12) that

$$0 \leqslant (-1)^j q_j^{(N)}(t) = (1-t)^m \delta_{0,j} + t^m \delta_{N-1,j}$$
$$+ \sum_{\mu=1}^{r} \frac{H_m(t, z_\mu)\,|z_\mu|^{N-j-1} + H_m(1-t, z_\mu)\,|z_\mu|^{j}}{H_m^z(0, z_\mu)(|z_\mu|^N + 1)}$$

holds for $t \in [0, 1]$ and $j \in \{0, 1, ..., N-1\}$. In the sum, each term is of the form

$$a x^{N-j-1} + b x^j,$$

where $a, b > 0$, $x > 1$. Hence it is a convex function with regard to j in $0 \leqslant j \leqslant N-1$. The same is true for the whole sum and also for the two preceding terms. Hence $(-1)^j q_j^{(N)}(t) = |q_j^{(N)}(t)|$ is convex for fixed $t \in [0, 1]$ with respect to j varying in $\{0, 1, ..., N-1\}$. This implies for $t \in [0, 1]$, $j \in \{1, ..., (N-1)/2\}$ the relation

$$|q_j^{(N)}(t)| \leqslant \max\{|q_{j-1}^{(N)}(t)|, |q_{N-j}^{(N)}(t)|\}$$
$$= \max\{|q_{j-1}^{(N)}(t)|, |q_{j-1}^{(N)}(1-t)|\};$$

compare (2). From this it follows by symmetry in t anf $1-t$ that, as well,

$$\max\{|q_j^{(N)}(t)|, |q_j^{(N)}(1-t)|\} \leqslant \max\{|q_{j-1}^{(N)}(t)|, |q_{j-1}^{(N)}(1-t)|\} \tag{17}$$

is valid. Especially we find that

$$\max\{|q_j^{(N)}(t)|: t \in [0, 1]\} \leqslant \max\{|q_{j-1}^{(N)}(t)|: t \in [0, 1]\} \tag{18}$$

holds for $j \in \{1, 2, ..., (N-1)/2\}$. Because of (4), the assertion of Theorem 2 is now a consequence of (12), and the theorem is proved.

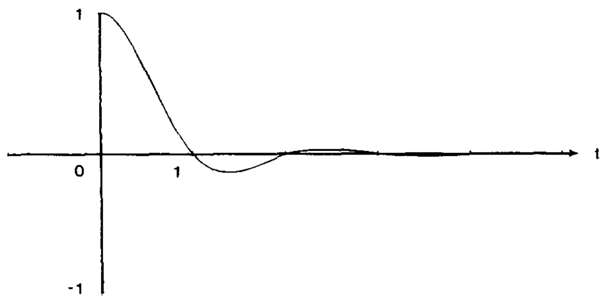

FIG. 1. Lagrangian cardinal spline $l_0^{(\infty)}$ according to (15).

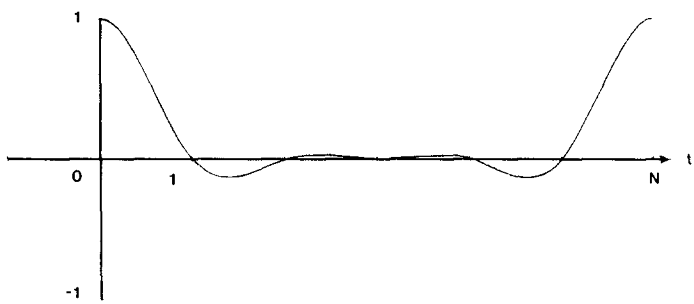

FIG. 2. Lagrangian periodic spline $l_0^{(N)}$ according to (16), if N is even.

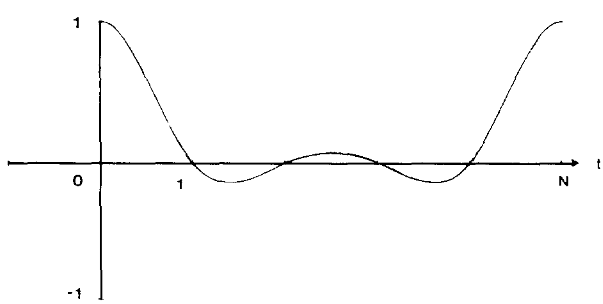

FIG. 3. Lagrangian periodic spline $l_0^{(N)}$ according to (18), if N is odd.

6. Qualitative Behaviour of Lagrangian Splines

Formulae (16) and (18), together with (15), give a more precise description of the qualitative behaviour of the periodic Lagrangian splines than has been given already by Richards [4]. Moreover, we obtained a corresponding result for the nonperiodic cardinal Lagrangian splines. The result is as could be expected from the usual polynomial interpolation; it is intuitively demonstrated in Figs. 1, 2, 3.

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